~~This dist. of time: functions of rvs, expectation~~

read: DS ch. 4
pp. 215 – 260

AMS131
18Aug17

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Case study: option pricing

~~This~~ ~~next~~ expectation time: of sums and products, variance, SD; other moments of the dist.

Silicon valley companies give signing bonuses as incentives to accept their job offers. These are often in the form of stock options: an opportunity to buy N shares of the company one year from now at a (known) price, S. If in your view the stock is likely to rise over the next year, you'll be able to sell at a profit. Define $X = \left(\begin{array}{c}\text{price of the}\\ \text{stock 1 year}\\ \text{from now}\end{array}\right)$.

For simplicity pretend X is discrete

with only 2 values: $x_1 < \$$ and (2)

$x_2 > \$$, let $p = \underline{P}(X = x_2)$, the prob.

that the stock will rise in value. You'd

like to evaluate these stock options

(e.g., to compare one company's job offer

with that of another), but (of course)

you don't know X. Let Y = value of option

for one share at $\$\$$ 1 year from now.

If ($X = x_1 < \$$), the option is worthless

and $Y = 0$; otherwise (ignoring dividends

& costs of buying & selling stocks) if

($X = x_2 > \$$) then the option is worth

$(x_2 - \$)$; thus $Y = h(X) = \begin{cases} 0 & \text{if } X = x_1 \\ (x_2 - \$) & x_2 \end{cases}$.

To see how valuable

the option is, you have to compare

it to the return you would have received (3)
if you had not "exercised the stock option";
a reasonable point of comparison would
be to invest in a bond or other fixed security that pays α%/year.

A fair measure of worth of the option
would be the <u>present value</u> of Y,
defined to be the number c such that

$E(Y) = (1+\alpha)\cdot c$. But we already know

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that $E(Y) = 0\cdot(1-p) + (x_2 - S)\cdot p = (x_2-S)p$,

so $(1+\alpha)\cdot c = (x_2 - S)\cdot p$ and $c = \left(\frac{x_2 - S}{1+\alpha}\right)\cdot p$.

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to finish the calculation you need to
specify p. The standard way to do this

in the financial sector is to assume that (4)

the present value of X is equal in expectation to the current value of the stock price: i.e., to assume that the expected value of (buying 1 share & holding it for a year) = (investing the same amount of money in the risk-free alternative) — i.e., $E(X) = (1+\alpha)\cdot S$.

But we already know that

$E(X) = p\cdot x_2 + (1-p)\cdot x_1 \stackrel{\Delta}{=} (1+\alpha)S$; solving for p gives $p = \dfrac{x_1 - (1+\alpha)S}{x_1 - x_2} = \dfrac{(1+\alpha)S - x_1}{x_2 - x_1}$.

So the fair price (c) of an option to buy one share is given by $c = \left(\frac{x_2 - S}{1+\alpha} \right) \left[\frac{(1+\alpha) S - x_1}{x_2 - x_1} \right]$. (5)

As we as illustration
$S = \$200$
$x_1 = \$180$
$x_2 = \$260$
$\alpha = .04$

downside $20 (−10%)

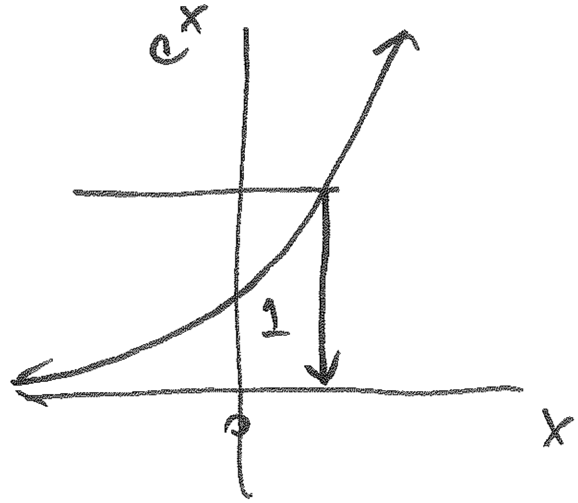
upside $60 (+30%)

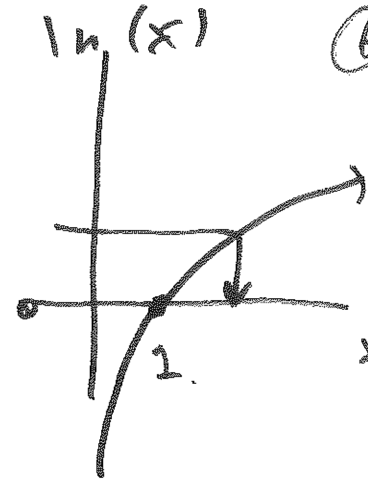
realistic in 2001 or so but not today: $\alpha \doteq .01$ now or .02

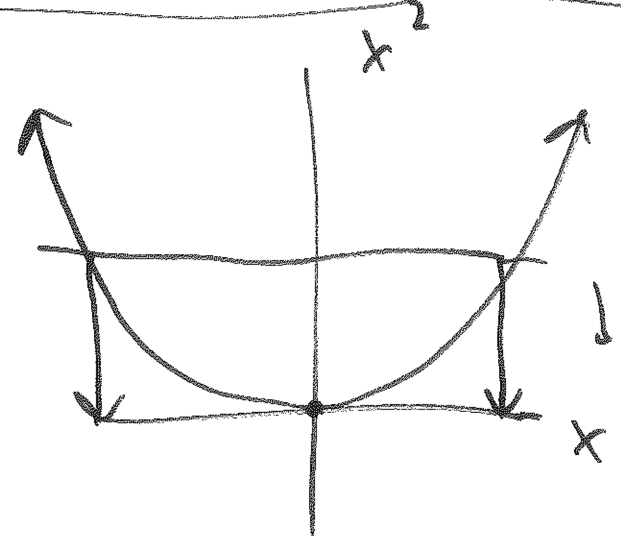
With these values $c \doteq \$20.19$ (about 10% of the current value of the stock). c is called the risk-neutral price of the option; under the assumptions made here, you could now sell the option today (if you had it) at a fair price of about $20; this would make you an options trader.

An investment that allows people to buy or sell an option on a security (e.g. stock) is called a derivative.

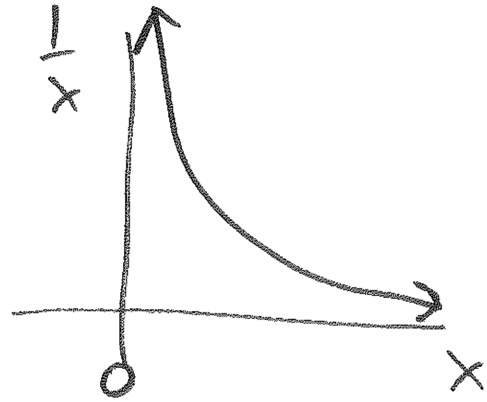
Examples of invertible (1-1) & differentiable functions

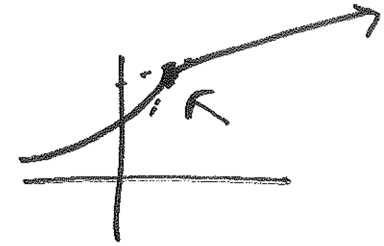




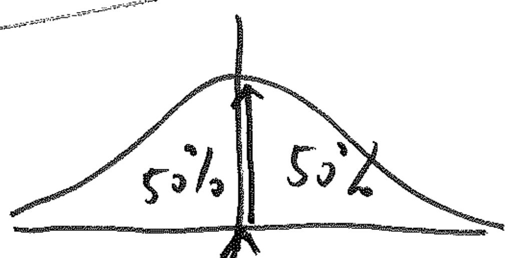


diff ✓
but not
1-1



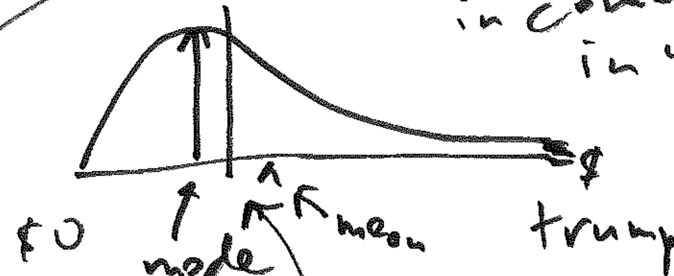


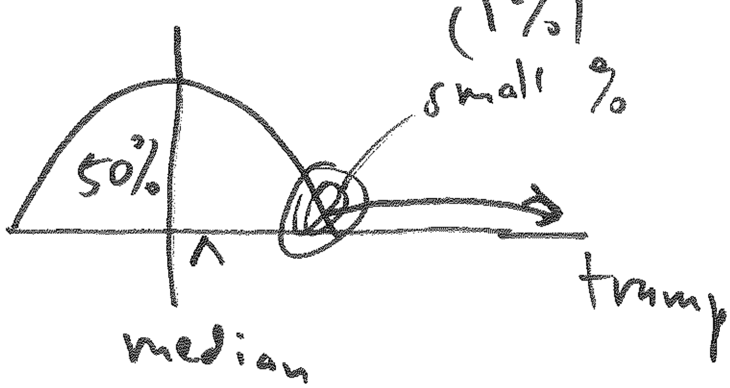
symmetric & unimodal (9.50)



point of symmetry
= mode
= median
= mean

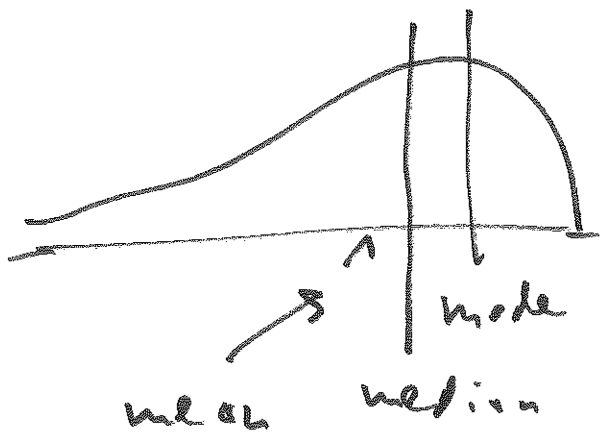
annual family income in U.S.

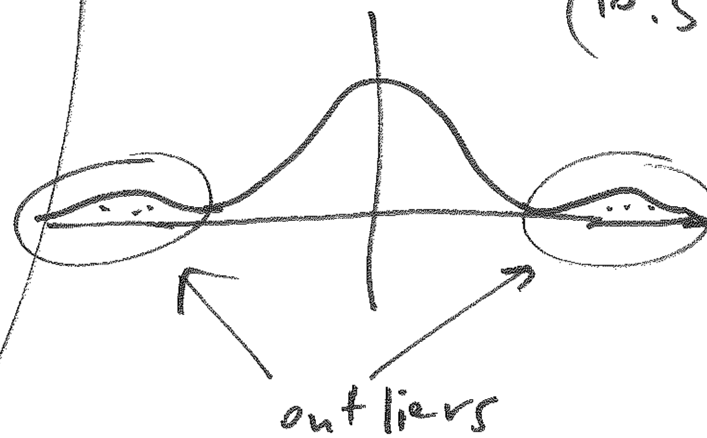




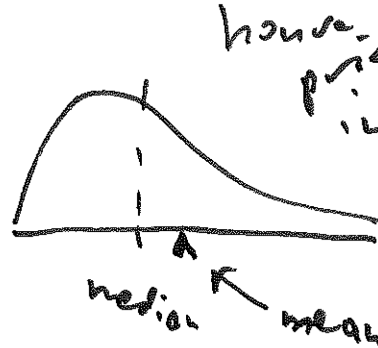
mean is pulled by the tail

(10.54)





mean much more influenced by outliers than median

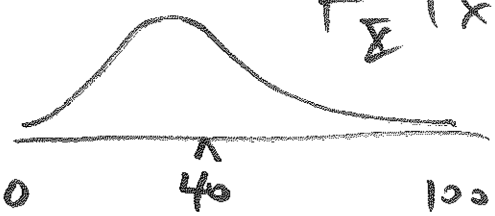


unscrupulous real estate person:

quote median to a buyer but mean to a seller

you: ask for both mean & median

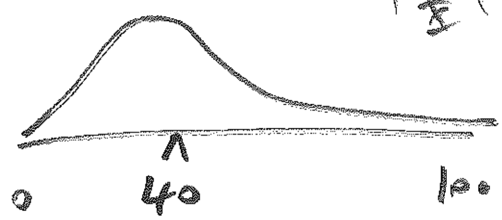
$f_X(x)$



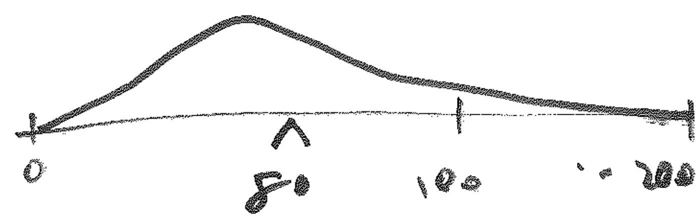
$Y = X + 10$

$E(Y) = E(X) + 10$

$f_X(x)$



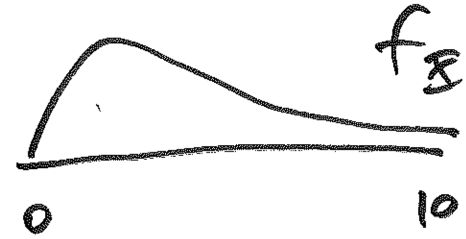
$Y = 2X$



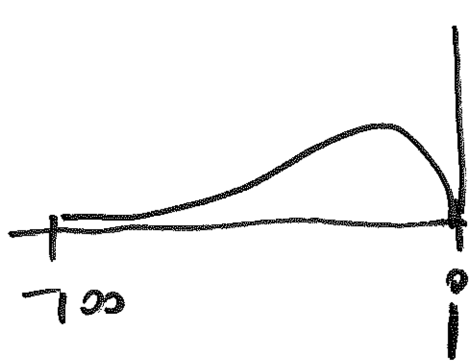
$E(Y) = 2E(X)$

$Y = aX + b \rightarrow E(Y) = aE(X) + b$

$f_X(x)$



$Y = -X$

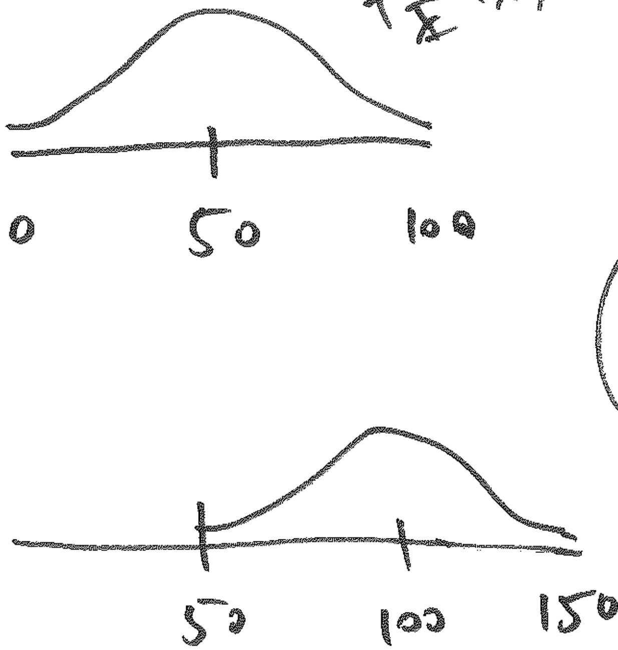


x

(9.55)(10.40)

PLAN AHEA

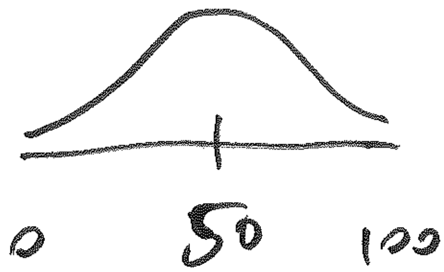
$f_X(x)$

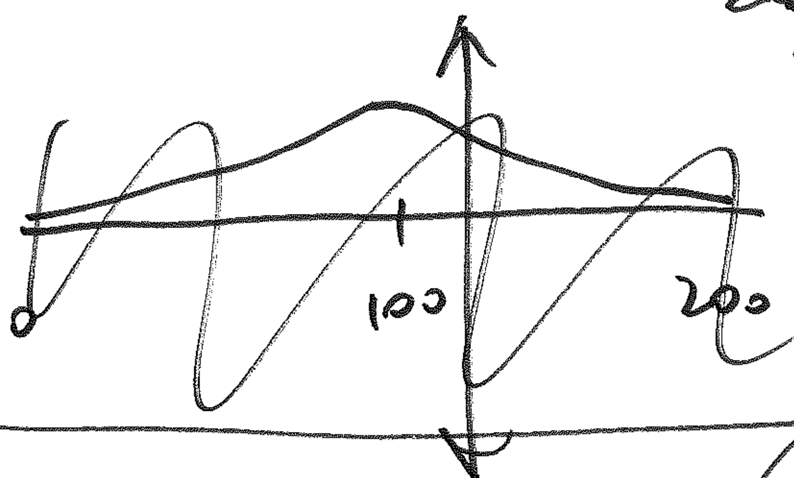


same spread & shape

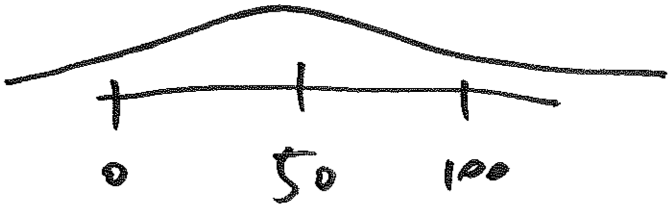
different center

$E(X)$

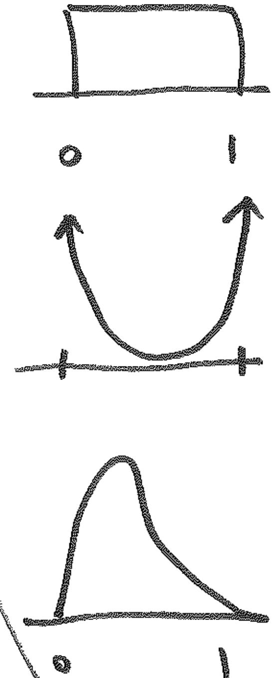




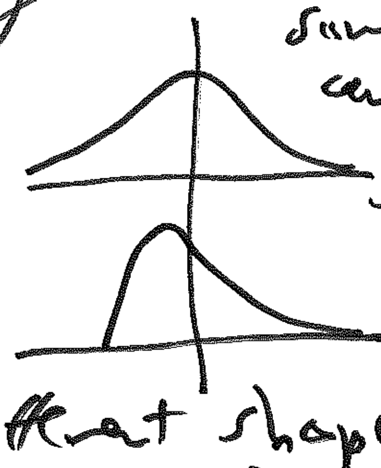
same center & shape, different spread



(range different)



same center & spread,



different shape

X_1, X_2 indep

$SD(X_1) = 2$

$SD(X_2) = 7$

$SD(X_1 + X_2) = \cancel{9}$

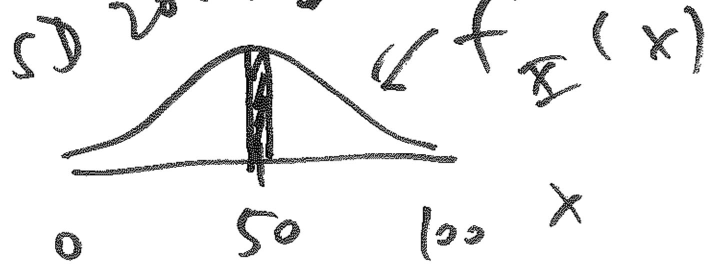
$SD(X_1 + X_2) = 11$

Empirical Rule

For (virtually) any dist'n of a r.v. X, if you start at the mean $\mu_X = E(X)$ and go $\left\{ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right\}$ SDs $\sigma_X = \sqrt{V(X)}$ either way, you will usually capture

$$\left\{\begin{array}{ll} \text{about } 2/3 & (68\%) \\ \text{most} & (95\%) \\ \text{almost all} & (99.7\%) \end{array}\right\} \text{ of the probability}$$

SD 20-25



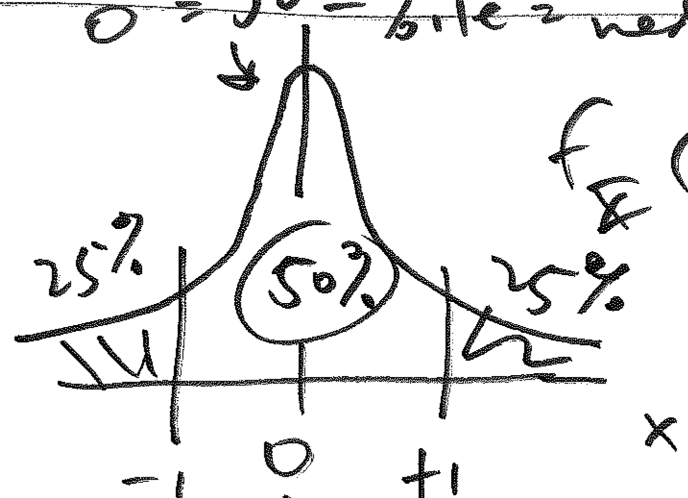
$E(X) = \mu_X$

$= 50$

$SD(X) = \sigma_X = \cancel{1}$ $(49, 51)$ too little prob.

$\cancel{25}$ $(-25, 125)$ too much

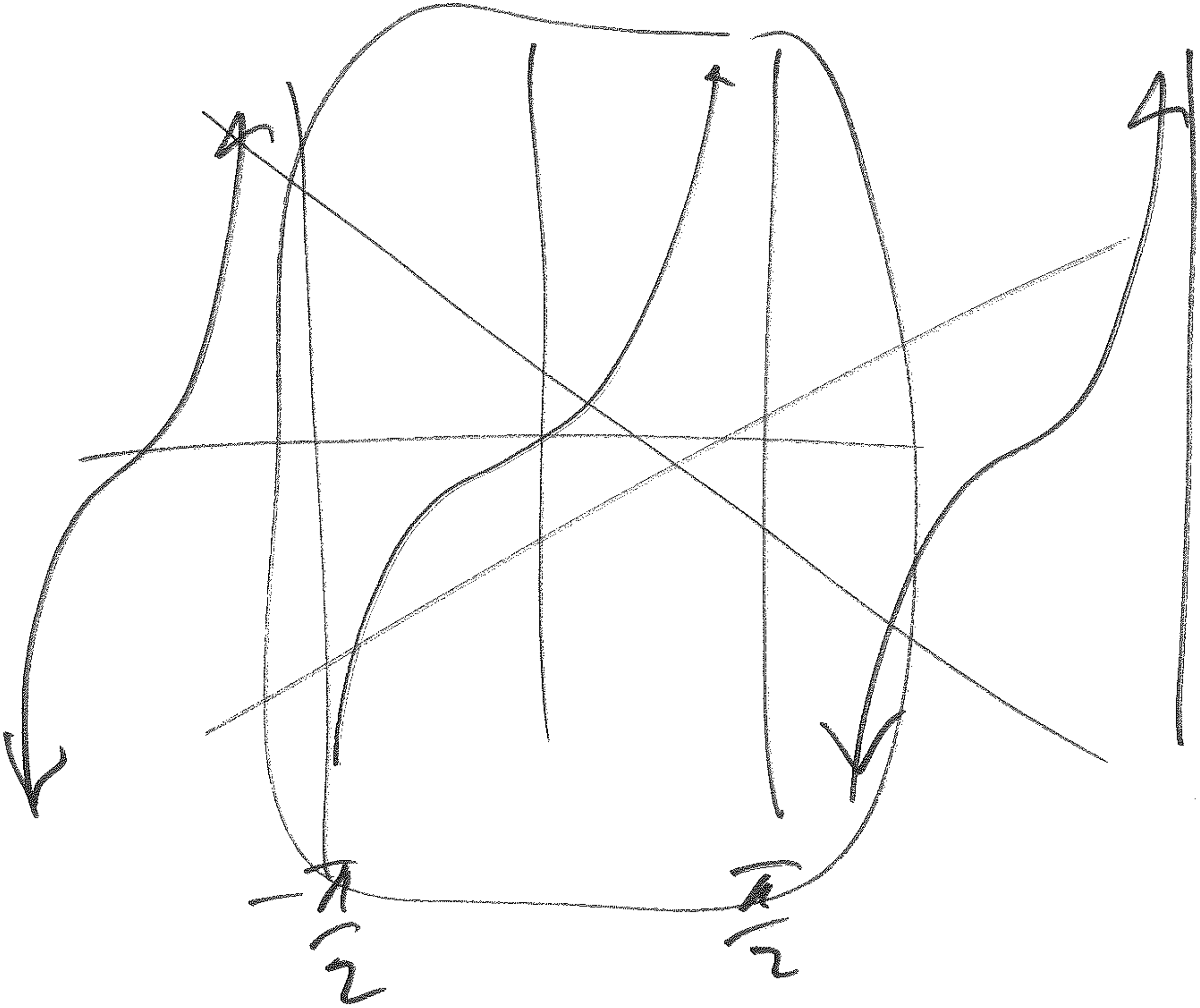
$0 = 50^{th}$ %ile = median



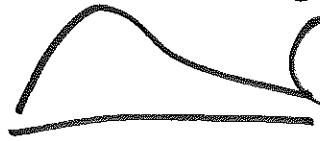
standard Cauchy

center = pt. of symmetry = 0

free symbolic: Wolfram alpha

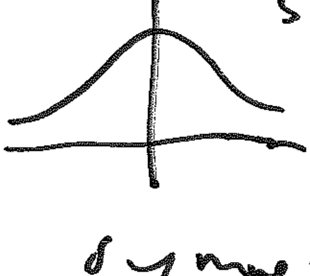


positively skewed

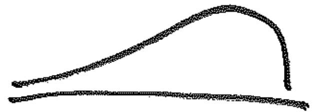


(skew > 0)

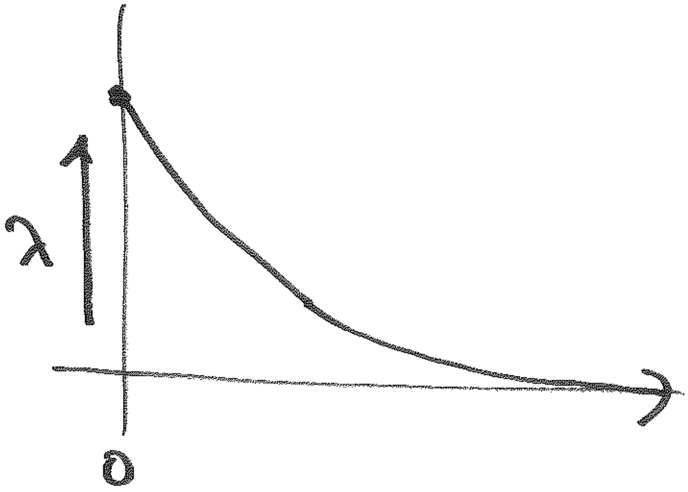
skewness 0



symmetric



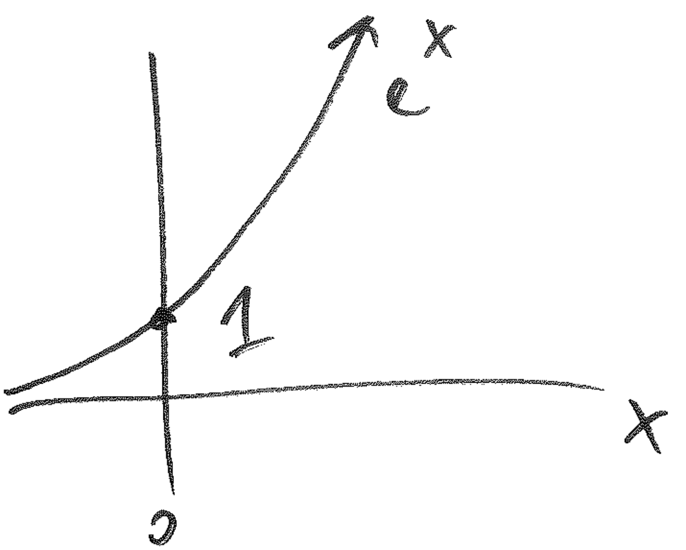
negatively skewed (skew < 0)

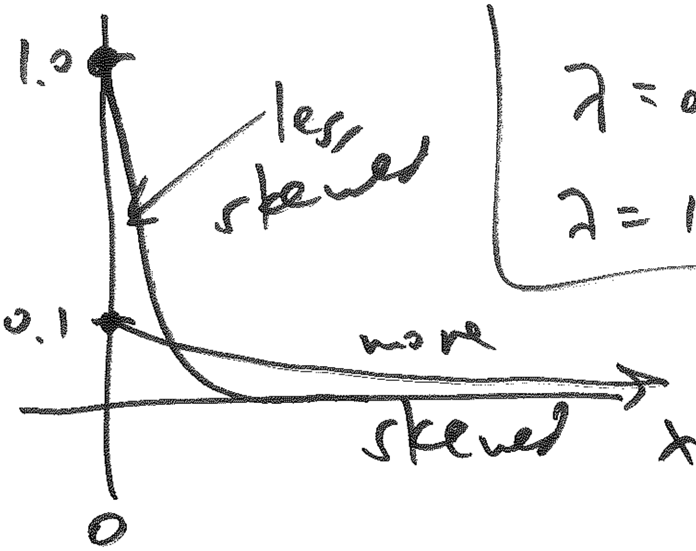


$$f_{X}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$

$$E(X) = \frac{1}{\lambda}$$

(positive skew)





$\lambda = 0.1$

$\lambda = 1$